

# The Andreka-Resek-Thompson and Ferenczi's results using games

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## Abstract

We provide a new proof of the celebrated Andréka-Resek-Thompson representability result of certain finitely axiomatized cylindric-like algebras, together with its quasipolyadic equality analogue proved by Ferenczi. Our proof uses games as introduced in algebraic logic by Hirsch and Hodkinson. Using the same method, we prove a new representability result for diagonal free reducts of such algebras. Such representability results provides completeness theorems for variants of first order logic, that can also be viewed as multi-modal logics. Finally, using a result of Marx, we show that all varieties considered enjoy the superamalgamation property, a strong form of amalgamation, implying that such logics also enjoy a Craig interpolation theorem.

## 1 Introduction

Stone's representation theorem for Boolean algebras can be formulated in two, essentially equivalent ways. Every Boolean algebra is isomorphic to a field of sets, or the class of Boolean set algebras can be axiomatized by a finite set of equations. As is well known, Boolean algebras constitute the algebraic counterpart of propositional logic. Stone's representation theorem, on the other hand, is the algebraic equivalent of the completeness theorem for propositional logic.

However, when we step inside the realm of first order logic, things tend to become more complicated. Not every abstract cylindric algebra is representable as a field of sets, where the extra Boolean operations of cylindrifiers and diagonal elements are faithfully represented by projections and equality. Disappointingly, the class of representable algebras fail to be axiomatized by any reasonable finite schema and its resistance to such axiomatizations is inevitable. This is basically a reflection of the essential incompleteness of natural (more basic) infinitary extensions of first order logic. In such extensions, unlike first order logic, validity cannot be captured by a finite schema.

Such extensions are obtained by dropping the condition of local finiteness (reflecting the simple fact that first order formulas contain only finitely many variables) in algebras considered, allowing formulas of infinite length. This is necessary if we want to deal with the so-called algebraizable extensions of first order logic; extensions that are akin to universal algebraic investigations.

The condition of local finiteness, cannot be expressed in first order logic, and this is not warranted if we want to deal, like in the case of Boolean algebras, only with equations, or at worst quasi-equations. Then we are faced with the following problem. Find a simple (hopefully finitary) axiomatization of classes of representable algebras abounding in algebraic logic, using only equations or quasi equations, which also means that we want to stay in the realm of quasivarieties.

There are two conflicting but complementary facets of such a problem, referred to in the literature, as the representation problem. One is to delve deeply in investigating the complexity of potential axiomatizations for existing varieties of representable algebras, the other is to try to sidestep such wild unruly complex axiomatizations, often referred to as *taming methods*. Those taming methods can either involve passing to (better behaved) expansions of the algebras considered, or else change the very notion of representability involved, as long as it remains concrete enough. The borderlines are difficult to draw, we do might not know what is *not* concrete enough, but we can judge that a given representability notion is satisfactory, once we have one. (This is analogous to undecidability issues, with the main difference that we do know what we mean by *not decidable*. We do not have an analogue of a 'recursive representability notion').

One of the taming methods is *relativization*, meaning that we search for representations on sets consisting of arbitrary  $\alpha$  sequences ( $\alpha$  an ordinal specifying the dimension of algebras considered), rather than squares, that is set of the form  ${}^\alpha U$  for some set  $U$ . It turns out that this can be done when do not insist on commutativity of cylindrifications. Dropping commutativity makes life much easier in many respects, not only representability. An example is decidability of the equational theory of the class of algebras in question. Typically given a set of equations  $\Sigma$ , show that  $\mathfrak{A} \models \Sigma$  iff  $\mathfrak{A}$  is representable as an algebra whose elements are genuine relations and operations are set theoretic operations pending only on manipulations of concrete relations. Very few positive results are known in this regard, the most famous is the Resek - Thompson celebrated theorem proved in [1]. The Resek - Thompson result is a refinement of a result of Resek due to Thompson. The first proof of Resek's result (that is slightly different) was more than 100 pages long. The short proof of the modified Resek - Thompson result in [1] is due to Andr  ka. In this paper we provide also a relatively short proof of this theorem using games as introduced in algebraic logic by Hirsch and Hodkinson [2].

In [7], Ferenczi talks about an important case for fields of sets that occur when the unit consists of a certain set of  $\alpha$ -sequences. In addition to the usual set theoretic boolean operations, the  $i$ th cylindrification and the constants  $i$ jth diagonal, new natural operations are imposed to describe such field of sets. Such operations are e.g. the elementary substitution  $[i|j]$  and the elementary transposition  $[i, j]$  for every  $i, j < \alpha$ , restricted to the unit. Ferenczi considers the extended field of sets which is closed under these operations and then gives a positive answer to the question: Do these fields of sets form a variety, and if so, what is its axiomatization?

Again, in this paper we provide a shorter proof of Ferenczi's result, which provides a *finite* axiomatization, using games as introduced in [2]. We follow the axiomatization provided by Ferenczi in his recent paper [7]. We refer the reader to [7] and [1] to get a grasp of the importance of this problem in algebraic logic.

Building representations can be implemented by the step-by-step method (as in [1] and [7]), which consists of treating defects one by one and then taking a limit where the contradictions disappear. What can be done by step-by-step constructions, can be done by games but not the other way round. Games were introduced in algebraic logic by Hirsch and Hodkinson. Such games, which are basically Banach-Mazur games in disguise, are games of infinite lengths between two players  $\forall$  and  $\exists$ . The real advantage of the game technique is that games do not only build representations, when we know that such representations exist, but they also tell us when such representations exist, if we do not know a priori that they do. The translation however from step-by step techniques to games is not always a purely mechanical process, even if we know that it can be done. This transfer can well involve some ingenuity, in obtaining games are transparent, intuitive and easy to grasp. It is an unsettled (philosophical) question as to which is more intuitive, step-by-step techniques or games. Basically this depends on the context, but in all cases it is nice to have both available if possible, when we know one exists. When we have a step-by-step technique, then we are sure that there is at least one corresponding game. Choosing a simple game is what counts at the end.

We follow the notation and terminology of [1] and [7]. In particular, for relations  $R, S$   $R|S = \{(a, b) : \exists c[(a, c) \in R, (c, b) \in S]\}$ .

**Definition 1.1.** *Class  $Crs_\alpha$*  An algebra  $\mathfrak{A}$  is a cylindric relativised set algebra of dimension  $\alpha$  with unit  $\mathbf{V}$  if it is of the form

$$\langle \mathbf{A}, \cup, \cap, \sim_{\mathbf{V}}, \phi, \mathbf{V}, \mathbf{C}_i^{\mathbf{V}}, \mathbf{D}_{ij}^{\mathbf{V}} \rangle_{i,j < \alpha},$$

where  $\mathbf{V}$  is a set of  $\alpha$ -termed sequences such that  $\mathbf{A}$  is a non-empty set of subsets of  $\mathbf{V}$ , closed under the Boolean operations  $\cup, \cap, \sim_{\mathbf{V}}$  and under the cylindrifications  $\mathbf{C}_i^{\mathbf{V}}X = \{y \in \mathbf{V} : y_u^i \in X \text{ for some } u\}$ , where  $i < \alpha$ ,  $X \in \mathbf{A}$ , and  $\mathbf{A}$  contain the elements  $\phi, \mathbf{V}$  and the diagonals  $\mathbf{D}_{ij}^{\mathbf{V}} = \{y \in \mathbf{V} : y_i = y_j\}$ .

**Class  $Drs_\alpha$**  An algebra  $\mathfrak{A}$  is a diagonal free relativised set algebra of dimension  $\alpha$  with unit  $\mathbf{V}$  if it is of the form

$$\langle \mathbf{A}, \cup, \cap, \sim_{\mathbf{V}}, \phi, \mathbf{V}, \mathbf{C}_i^{\mathbf{V}} \rangle_{i,j < \alpha},$$

where  $\mathbf{V}$  is a set of  $\alpha$ -termed sequences such that  $\mathbf{A}$  is a non-empty set of subsets of  $\mathbf{V}$ , closed under the Boolean operations  $\cup, \cap, \sim_{\mathbf{V}}$  and under the cylindrifications  $\mathbf{C}_i^{\mathbf{V}}X = \{y \in \mathbf{V} : y_u^i \in X \text{ for some } u\}$ , where  $i < \alpha$ ,  $X \in \mathbf{A}$ , and  $\mathbf{A}$  contain the elements  $\phi$  and  $\mathbf{V}$ .

The meaning of the notation  $y_u^i$  is  $(y_u^i)_j = y_j$  if  $j \neq i$  and  $(y_u^i)_j = u$  if  $j = i$ .  
**Some concepts and notation concerning  $Crs_\alpha$  and  $Drs_\alpha$ :**

- The class  $Crs_\alpha$  is a subclass of  $Drs_\alpha$ .
- Let  $\mathfrak{A} \in Crs_\alpha$  be with unit element  $\mathbf{V}$ . The relativized substitution operator<sup>1</sup>  ${}^{\mathbf{V}}S_j^i$  is defined as

$${}^{\mathbf{V}}S_j^i X = \mathbf{C}_i^{\mathbf{V}}(\mathbf{D}_{ij}^{\mathbf{V}} \cap X) \quad (X \in \mathbf{A}).$$

We often omit the superscript  $\mathbf{V}$  from  ${}^{\mathbf{V}}S_j^i$  and write  $S_j^i$ .

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<sup>1</sup>Clearly  $\mathbf{A}$  is closed under  ${}^{\mathbf{V}}S_j^i$ , since  $\mathbf{A}$  is closed under the cylindrifications and contains the diagonals.

- The transformation  $\tau$  defined on  $\alpha$  is called finite if  $\tau i = i$  except for finitely many  $i < \alpha$ . Important special cases of the finite transformations are the transformations  $[i|j]$ , called elementary substitutions, and  $[i, j]$ , called transpositions.
- Let  $\mathfrak{A} \in Drs_\alpha$  be with unit element  $V$ . If  $y \in V$  and  $\tau$  is any finite transformation on  $\alpha$ ,  ${}^V S_\tau y$  is defined as  $\tau|y$ . If  $X \in A$ , then  ${}^V S_\tau X$  is defined as  $\{\tau|y : y \in X\}^2$ .

**Definition 1.2** (Class  $D_\alpha$ ). It is the subclass of  $Crs_\alpha$  for which  ${}^V S_j^i V = V$  for every  $i, j < \alpha$ , where  $V$  is the unit of the algebra.

If we consider the substitution operators  ${}^V S_{[i|j]}$  and the transposition operators  ${}^V S_{[i, j]}$  defined above, then we get the polyadic versions of  $Crs_\alpha$ .

**Definition 1.3.** Class  $Prs_\alpha$  An algebra

$$\mathfrak{B} = \langle A, \cup, \cap, \sim_V, \phi, V, C_i^V, {}^V S_{[i|j]}, {}^V S_{[i, j]} \rangle_{i, j < \alpha}$$

is an  $\alpha$ -dimensional polyadic relativized set algebra if for the diagonal free reduct,  $\mathfrak{Rd}_{df} \mathfrak{B} \in Drs_\alpha$ ; further,  $A$  is closed under the transpositions  ${}^V S_{[i, j]}$  and the substitutions  ${}^V S_{[i|j]}$ .

**Class**  $Pers_\alpha$  An algebra

$$\mathfrak{B} = \langle A, \cup, \cap, \sim_V, \phi, V, C_i^V, {}^V S_{[i|j]}, {}^V S_{[i, j]}, D_{ij}^V \rangle_{i, j < \alpha}$$

is an  $\alpha$ -dimensional polyadic equality relativized set algebra if for the cylindric reduct,  $\mathfrak{Rd}_{ca} \mathfrak{B} \in Crs_\alpha$ ; further,  $A$  is closed under the transpositions  ${}^V S_{[i, j]}^3$ .

**Class**  $Srs_\alpha$  An algebra

$$\mathfrak{B} = \langle A, \cup, \cap, \sim_V, \phi, V, C_i^V, {}^V S_{[i|j]} \rangle_{i, j < \alpha}$$

is an  $\alpha$ -dimensional substitution relativized set algebra if for the diagonal free reduct,  $\mathfrak{Rd}_{df} \mathfrak{B} \in Drs_\alpha$ ; further,  $A$  is closed under the substitutions  ${}^V S_{[i|j]}$ .

**Definition 1.4** (Class  $Dp_\alpha$ ). It is the subclass of  $Prs_\alpha$  for which  ${}^V S_{[i|j]} V = V$  for every  $i, j < \alpha$ , where  $V$  is the unit of the algebra.

**Definition 1.5** (Class  $Dpe_\alpha$ ). It is the subclass of  $Pers_\alpha$  for which  ${}^V S_{[i|j]} V = V$  for every  $i, j < \alpha$ , where  $V$  is the unit of the algebra.

**Definition 1.6** (Class  $Ds_\alpha$ ). It is the subclass of  $Srs_\alpha$  for which  ${}^V S_{[i|j]} V = V$  for every  $i, j < \alpha$ , where  $V$  is the unit of the algebra.

We assume the knowledge of the concepts of cylindric algebras [3]. The cylindric axiom

$$(C_4) \quad c_i c_j x = c_j c_i x$$

<sup>2</sup>A need not to be closed under  ${}^V S_\tau$  for any arbitrary  $\tau$ .

<sup>3</sup>Clearly,  $A$  is closed under the substitutions  ${}^V S_{[i|j]}$  because  ${}^V S_{[i|j]} = {}^V S_j^i$ .

proved to be quite a strong property. In the following axiomatization this property is replaced by a weaker property. Furthermore, what is called merry-go-round axioms (**MGR**) are postulated. By Resek-Thompson theorem [1], the existence of such axioms yields representability by relativized set algebra. The following axiomatization is due to Thompson and Andreka.

**Definition 1.7** (Class  $PTA_\alpha$ ). An algebra  $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j \in \alpha}$ , where  $+, \cdot$  are binary operations,  $-, c_i$  are unary operations and  $0, 1, d_{ij}$  are constants for every  $i, j \in \alpha$ , is partial transposition algebra<sup>4</sup> if it satisfies the following identities for every  $i, j, k \in \alpha$ .

( $C_0$ ) – ( $C_3$ )  $\langle A, +, \cdot, -, 0, 1, c_i \rangle_{i \in \alpha}$  is a Boolean algebra with additive closure operators  $c_i$  such that the complements of  $c_i$ -closed elements are  $c_i$ -closed,

( $C_4$ )\*  $c_i c_j x \geq c_j c_i x \cdot d_{jk}$  if  $k \notin \{i, j\}$ ,

( $C_5$ )  $d_{ii} = 1$ ,

( $C_6$ )  $d_{ij} = c_k(d_{ik} \cdot d_{kj})$  if  $k \notin \{i, j\}$ ,

( $C_7$ )  $d_{ij} \cdot c_i(d_{ij} \cdot x) \leq x$  if  $i \neq j$ ,

(**MGR**) for every  $i, j \in \alpha$ ,  $i \neq j$ , let  $s_j^i x = c_i(d_{ij} \cdot x)$ ,  $s_i^i x = x$ . Then:

$$s_i^k s_j^i s_m^j c_k x = s_m^k s_i^m s_j^i c_k x \text{ if } k \notin \{i, j, m\}, m \notin \{i, j\}$$

We also assume the basic knowledge of the concepts of polyadic equality and quasi-polyadic equality algebras [4, p. 266]. The so called finitary polyadic equality algebras, i.e., the class  $FPEA_\alpha$ , is term definitionally equivalent to the quasi-polyadic equality algebras [10, Theorem 1]. The axiomatization of  $FPEA_\alpha$  and of the class  $TEA_\alpha$  to be introduced are different in only one axiom, namely the axiom

( $F_5$ )  $s_j^i c_k x = c_k s_j^i x$  if  $k \notin \{i, j\}$ .

The following axiomatization is due to Ferenczi, abstracting away from the class  $Pers_\alpha$  (meaning that the axioms all hold in  $Pers_\alpha$ ). This is a soundness condition. Ferenczi proves completeness of these axioms, which we also prove using the different technique of resorting to games.

**Definition 1.8** (Class  $TEA_\alpha$ ). A transposition equality algebra of dimension  $\alpha$  is an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i, s_{ij}, d_{ij} \rangle_{i,j \in \alpha},$$

where  $c_i, s_j^i, s_{ij}$  are unary operations,  $d_{ij}$  are constants, the axioms ( $F_0$ )-( $F_9$ ) below are valid for every  $i, j, k < \alpha$ :

( $Fe_0$ )  $\langle A, +, \cdot, -, 0, 1 \rangle$  is a boolean algebra,  $s_i^i = s_{ii} = d_{ii} = Id \upharpoonright A$  and  $s_{ij} = s_{ji}$ ,

( $Fe_1$ )  $x \leq c_i x$ ,

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<sup>4</sup>We call it partial transposition algebra to illustrate that MGR axiom gives partial transpositions. This class is different from the class of partial transposition algebras defined in [6].

- (Fe<sub>2</sub>)  $c_i(x + y) = c_i x + c_i y$ ,
- (Fe<sub>3</sub>)  $s_j^i c_i x = c_i x$ ,
- (Fe<sub>4</sub>)  $c_i s_j^i x = s_j^i x$ ,  $i \neq j$ ,
- (Fe<sub>5</sub>)<sup>\*</sup>  $s_j^i s_m^k x = s_m^k s_j^i x$  if  $i, j \notin \{k, m\}$ ,
- (Fe<sub>6</sub>)  $s_j^i$  and  $s_{ij}$  are boolean endomorphisms,
- (Fe<sub>7</sub>)  $s_{ij} s_{ij} x = x$ ,
- (Fe<sub>8</sub>)  $s_{ij} s_{ik} x = s_{jk} s_{ij} x$ ,  $i, j, k$  are distinct,
- (Fe<sub>9</sub>)  $s_{ij} s_j^i x = s_i^j x$ ,
- (Fe<sub>10</sub>)  $s_j^i d_{ij} = 1$ ,
- (Fe<sub>11</sub>)  $x \cdot d_{ij} \leq s_j^i x$ .

For  $\mathfrak{A} \in TA_\alpha$ , its partial transposition reduct is the structure

$$\mathfrak{A}_{pt} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i, d_{ij} \rangle_{i,j \in \alpha}.$$

The following axiomatization is new, it is obtained from Ferenczi's axiomatization by dropping equations involving diagonal elements.

**Definition 1.9** (Class  $TA_\alpha$ ). *A transposition algebra of dimension  $\alpha$  is an algebra*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i, s_{ij} \rangle_{i,j \in \alpha},$$

where  $c_i, s_j^i, s_{ij}$  are unary operations, the axioms (F<sub>0</sub>)-(F<sub>9</sub>) below are valid for every  $i, j, k < \alpha$ :

- (F<sub>0</sub>)  $\langle A, +, \cdot, -, 0, 1 \rangle$  is a boolean algebra,  $s_i^i = s_{ii} = d_{ii} = Id \upharpoonright A$  and  $s_{ij} = s_{ji}$ ,
- (F<sub>1</sub>)  $x \leq c_i x$ ,
- (F<sub>2</sub>)  $c_i(x + y) = c_i x + c_i y$ ,
- (F<sub>3</sub>)  $s_j^i c_i x = c_i x$ ,
- (F<sub>4</sub>)  $c_i s_j^i x = s_j^i x$ ,  $i \neq j$ ,
- (F<sub>5</sub>)<sup>\*</sup>  $s_j^i s_m^k x = s_m^k s_j^i x$  if  $i, j \notin \{k, m\}$ ,
- (F<sub>6</sub>)  $s_j^i$  and  $s_{ij}$  are boolean endomorphisms,
- (F<sub>7</sub>)  $s_{ij} s_{ij} x = x$ ,
- (F<sub>8</sub>)  $s_{ij} s_{ik} x = s_{jk} s_{ij} x$ ,  $i, j, k$  are distinct,
- (F<sub>9</sub>)  $s_{ij} s_j^i x = s_i^j x$ .

The following axiomatization is new. It is similar to Pinter's axiomatization [12] with two major differences. We do not have commutativity of cylindrifications, this is one thing; the other is that we stipulate the *MGR* identities.

**Definition 1.10** (Class  $SA_\alpha$ ). A substitution algebra of dimension  $\alpha$  is an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i \rangle_{i,j \in \alpha},$$

where  $c_i, s_j^i$  are unary operations, the axioms  $(S_0)$ – $(F_8)$  below are valid for every  $i, j, k < \alpha$ :

$(S_0)$   $\langle A, +, \cdot, -, 0, 1 \rangle$  is a boolean algebra and  $s_i^i = Id \upharpoonright A$ ,

$(S_1)$   $x \leq c_i x$ ,

$(S_2)$   $c_i(x + y) = c_i x + c_i y$ ,

$(S_3)$   $s_j^i c_i x = c_i x$ ,

$(S_4)$   $c_i s_j^i x = s_j^i x$ ,  $i \neq j$ ,

$(S_5)^*$   $s_j^i s_m^k x = s_m^k s_j^i x$  if  $i, j \notin \{k, m\}$ ,

$(S_6)$   $s_j^i$  is boolean endomorphism,

$(S_7)$   $s_k^k s_i^j x = s_i^j s_k^k x$ ,

$(S_8)$

$$s_i^k s_j^i s_m^j c_k x = s_m^k s_i^m s_j^i c_k x \text{ if } k \notin \{i, j, m\}, m \notin \{i, j\}$$

$(F_5)^*$  (and also  $(F_5)^*$  and  $(S_5)^*$ ) is obviously a weakening of  $(F_5)$ . Also it is known that  $\mathfrak{A} \mathfrak{D}_{pt} \mathfrak{A} \in PTA_\alpha$ , for any  $\mathfrak{A} \in TEA_\alpha$  [10]. We consider as known the concept of the substitution operator  $s_\tau$  defined for any finite transformation  $\tau$  on  $\alpha$ ;  $s_\tau$  can be introduced uniquely in  $FPEA_\alpha$  and in  $TEA_\alpha$ , too. The existence of such an  $s_\tau$  follows from the proof of [10, Theorem 1(ii)], it is easy to check that the proof works by assuming  $(F_5)^*$  instead of  $(F_5)$  and (notationally) the composition operator  $|$  instead of  $\circ$ .

Throughout this paper we assume that the polyadic-like algebras occurring here are equipped with the operator  $s_\tau$ , where  $\tau$  is finite. Further,  $s_\tau$  is assumed to have the following properties for arbitrary finite transformations  $\tau$  and  $\lambda$  and ordinals  $i, j < \alpha$  (by [10, p.542]):

$$s_{\tau|\lambda} = s_\tau s_\lambda^5,$$

$$s_j^i = s_{[i|j]},$$

$$s_\tau d_{ij} = d_{\tau i \tau j} \text{ (of course only in the class } TEA_\alpha),$$

$$c_i s_\tau \leq s_\tau c_{\tau^{-1}}, \text{ here } \tau \text{ is finite permutation.}$$

**Lemma 1.1.** Let  $\alpha$  be an ordinal,  $\mathfrak{A} \in TEA_\alpha$ ,  $a \in \text{At}\mathfrak{A}$  and  $i, j \in \alpha$ . Then

$$a \leq d_{ij} \implies s_{[i,j]} a = a.$$

*Proof.* See [7, p. 875]. □

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<sup>5</sup>This depends on  $(F_7)$  and  $(F_8)$ .

## 2 Games and Networks

In this section fix  $n \in \omega$ . We start with some preparations. Let  $\bar{x}, \bar{y}$  be  $n$ -tuples of elements of some set. We write  $x_i$  for the  $i$ th element of  $\bar{x}$ , for  $i < n$ , so that  $\bar{x} = (x_0, \dots, x_{n-1})$ . For  $i < n$ , we write  $\bar{x} \equiv_i \bar{y}$  if  $x_j = y_j$  for all  $j < n$  with  $j \neq i$ . The next two definitions are taken from [2]. A (relativized) network is a finite approximation to a (relativized) representation.

**Definition 2.1.**

- Let  $\mathfrak{A} \in PTA_n$ . A relativized  $\mathfrak{A}$  pre-network is a pair  $N = (N_1, N_2)$  where  $N_1$  is a finite set of nodes  $N_2 : N_1^n \rightarrow \mathfrak{A}$  is a partial map, such that if  $f \in \text{dom} N_2$ , and  $i, j < n$  then  $f_{f(j)}^i \in \text{Dom} N_2$ .  $N$  is atomic if  $\text{Range} N \subseteq \text{At} \mathfrak{A}$ . We write  $N$  for any of  $N, N_1, N_2$  relying on context, we write  $\text{nodes}(N)$  for  $N_1$  and  $\text{edges}(N)$  for  $\text{dom}(N_2)$ .  $N$  is said to be a network if
  - (a) for all  $\bar{x} \in \text{edges}(N)$ , we have  $N(\bar{x}) \leq d_{ij}$  iff  $x_i = x_j$ .
  - (b) if  $\bar{x} \equiv_i \bar{y}$ , then  $N(\bar{x}) \cdot c_i N(\bar{y}) \neq 0$ .
- Let  $\mathfrak{A} \in TEA_n$ . A relativized  $\mathfrak{A}$  pre-network is a pair  $N = (N_1, N_2)$  where  $N_1$  is a finite set of nodes  $N_2 : N_1^n \rightarrow \mathfrak{A}$  is a partial map, such that if  $f \in \text{dom} N_2$ , and  $\tau$  is a finite transformation then  $\tau|f \in \text{Dom} N_2$ . Again  $N$  is atomic if  $\text{Range} N \subseteq \text{At} \mathfrak{A}$ . Also we write  $N$  for any of  $N, N_1, N_2$  relying on context, we write  $\text{nodes}(N)$  for  $N_1$  and  $\text{edges}(N)$  for  $\text{dom}(N_2)$ .  $N$  is said to be a network if
  - (a) for all  $\bar{x} \in \text{edges}(N)$ , we have  $N(\bar{x}) \leq d_{ij}$  iff  $x_i = x_j$ ,
  - (b) if  $\bar{x}, \bar{y} \in \text{edges}(N)$  and  $\bar{x} \equiv_i \bar{y}$ , then  $N(\bar{x}) \cdot c_i N(\bar{y}) \neq 0$ ,
  - (c)  $N([i, j]|\bar{x}) = s_{[i, j]} N(\bar{x})$ , for all  $\bar{x} \in \text{edges}(N)$  and all  $i, j < n$ .

**Definition 2.2.** Let  $\mathfrak{A} \in PTA_n \cup TEA_n$ . We define a game denoted by  $G_\omega(\mathfrak{A})$  with  $\omega$  rounds, in which the players  $\forall$  (male) and  $\exists$  (female) build an infinite chain of relativized  $\mathfrak{A}$  pre-networks

$$\emptyset = N_0 \subseteq N_1 \subseteq \dots$$

In round  $t$ ,  $t < \omega$ , assume that  $N_t$  is the current prenetwork, the players move as follows:

- (a)  $\forall$  chooses a non-zero element  $a \in \mathfrak{A}$ ,  $\exists$  must respond with a relativized prenetwork  $N_{t+1} \supseteq N_t$  containing an edge  $e$  with  $N_{t+1}(e) \leq a$ ,
- (b)  $\forall$  chooses an edge  $\bar{x}$  of  $N_t$  and an element  $a \in \mathfrak{A}$ .  $\exists$  must respond with a pre-network  $N_{t+1} \supseteq N_t$  such that either  $N_{t+1}(\bar{x}) \leq a$  or  $N_{t+1}(\bar{x}) \leq -a$ ,
- (c) or  $\forall$  may choose an edge  $\bar{x}$  of  $N_t$  an index  $i < n$  and  $b \in \mathfrak{A}$  with  $N_t(\bar{x}) \leq c_i b$ .  $\exists$  must respond with a prenetwork  $N_{t+1} \supseteq N_t$  such that for some  $z \in N_{t+1}$ ,  $N_{t+1}(\bar{x}_z^i) = b$ .

$\exists$  wins if each relativized pre-network  $N_0, N_1, \dots$  played during the game is actually a relativized network. Otherwise,  $\forall$  wins. There are no draws.



Here we follow closely Hirsch-Hodkinson's techniques adapted to the present situation.

**Lemma 2.1.** *Let  $\mathfrak{A} \in PTA_n$  be atomic. For all  $i, j \in n$ ,  $i \neq j$ , define  $\mathfrak{t}_j^i x = \mathfrak{d}_{ij} \cdot \mathfrak{c}_i x$  and  $\mathfrak{t}_i^i x = x$ . Then*

$$(i) \ (\mathfrak{t}_j^i)^{\mathfrak{A}} : \text{At}\mathfrak{A} \rightarrow \text{At}\mathfrak{A}$$

(ii) *Let  $\Omega = \{\mathfrak{t}_i^j : i, j \in n\}^*$ , where for any set  $H$ ,  $H^*$  denotes the free monoid generated by  $H$ . Let*

$$\sigma = \mathfrak{t}_{j_1}^{i_1} \dots \mathfrak{t}_{j_n}^{i_n}$$

*be a word. Then define for  $a \in A$ :*

$$\sigma^{\mathfrak{A}}(a) = (\mathfrak{t}_{j_1}^{i_1})^{\mathfrak{A}}((\mathfrak{t}_{j_2}^{i_2})^{\mathfrak{A}} \dots (\mathfrak{t}_{j_n}^{i_n})^{\mathfrak{A}}(a) \dots),$$

*and*

$$\hat{\sigma} = [i_1|j_1][i_2|j_2] \dots [i_n|j_n].$$

*Then*

$$\mathfrak{A} \models \sigma(x) = \tau(x) \text{ if } \hat{\sigma} = \hat{\tau}, \sigma, \tau \in \Omega.$$

*That is for all  $\sigma, \tau \in \Omega$ , if  $\hat{\sigma} = \hat{\tau}$ , then for all  $a \in A$ , we have  $\sigma^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}}(a)$ .*

*Proof.* cf. [1] proof of Lemma 1 therein. The **MGR**, merry go round identities are used here. We note that Andreka's proof of this lemma is long, but using fairly obvious results on semigroups a much shorter proof can be given.  $\square$

**Lemma 2.2.** *Let  $\mathfrak{A} \in PTA_n$  be atomic. For all  $i, j \in n$ ,  $i \neq j$ , let  $\mathfrak{t}_j^i x$  be as above. The following hold for all  $i, j, k, l \in n$ :*

$$(i) \ (\mathfrak{t}_j^i)^{\mathfrak{A}} x \leq \mathfrak{d}_{ij} \text{ for all } x \in \mathfrak{A}.$$

$$(ii) \ (x \leq \mathfrak{d}_{ij} \Rightarrow (\mathfrak{t}_i^k)^{\mathfrak{A}} x \leq \mathfrak{d}_{ij} \cdot \mathfrak{d}_{ik} \cdot \mathfrak{d}_{jk}) \text{ for all } x \in \mathfrak{A}.$$

$$(iii) \ (a \leq \mathfrak{c}_i b \Leftrightarrow \mathfrak{c}_i a = \mathfrak{c}_i b) \text{ for all } a, b \in \text{At}\mathfrak{A}.$$

$$(iv) \ \mathfrak{c}_i(\mathfrak{t}_j^i)^{\mathfrak{A}} x = \mathfrak{c}_i x \text{ for all } x \in \mathfrak{A}.$$

*Proof.* (i) Follows directly from the definition of  $(\mathfrak{t}_j^i)^{\mathfrak{A}}$ .

(ii) First we need to check the following for all  $i, j, k < n$ :

- $\mathfrak{c}_k \mathfrak{d}_{ij} = \mathfrak{d}_{ij}$  if  $k \notin \{i, j\}$ . For, see [3, Theorem 1.3.3], the proof doesn't involve  $(C_4)$ .
- $\mathfrak{d}_{ij} = \mathfrak{d}_{ji}$ . For, see [3, Theorem 1.3.1], the proof works, indeed it doesn't depend on  $(C_4)$ .
- $\mathfrak{d}_{ij} \cdot \mathfrak{d}_{jk} = \mathfrak{d}_{ij} \cdot \mathfrak{d}_{ik}$ . A proof for such can be founded in [3, Theorem 1.3.7].

Now we can proof (ii):

$$\begin{aligned}
t_i^k x &= d_{ik} \cdot c_k x \\
&\leq d_{ik} \cdot c_k d_{ij} \\
&= d_{ik} \cdot d_{ij} \\
&= d_{ki} \cdot d_{ij} \\
&= d_{ki} \cdot d_{kj}
\end{aligned}
\tag{1}$$

$$= d_{ki} \cdot d_{kj} \tag{2}$$

From (1), (2) the desired follows.

(iii) – (iv) See [1, p. 675-676]

□

**Definition 2.3** (Partial transposition network). *Let  $\mathfrak{A}$  be an atomic  $PTA_n$  and fix an atom  $a \in \text{At}\mathfrak{A}$ . Let  $\bar{x}$  be any  $n$ -tuple (of nodes) such that  $x_i = x_j$  if and only if  $a \leq d_{ij}$  for all  $i, j < n$ . Let  $NSQ_{\bar{x}} = \{\bar{y} \in {}^n\{x_0, x_1, \dots, x_{n-1}\} : |\text{Range}(\bar{y})| < n\}$  ( $NS$  stands for non-surjective sequences). We define the partial transposition network  $PT_{\bar{x}}^{(a)} : NSQ_{\bar{x}} \rightarrow \text{At}\mathfrak{A}$  As follows: If  $\bar{y} \in NSQ_{\bar{x}}$ , then  $\bar{y} = [i_0|j_0][i_1|j_1] \cdots [i_k|j_k]|\bar{x}$ , for some  $i_0, i_1, \dots, i_k, j_0, j_1, \dots, j_k < n$ . Let  $PT_{\bar{x}}^{(a)}(\bar{y}) = (t_{i_0}^{j_0} \cdots t_{i_k}^{j_k})^{(\mathfrak{A})}a$ . This is well defined by Lemma 2.1*

**Definition 2.4** (Transposition network). *Let  $\mathfrak{A}$  be an atomic  $TEA_n$  and fix an atom  $a \in \text{At}\mathfrak{A}$ . Let  $\bar{x}$  be any  $n$ -tuple of nodes such that  $x_i = x_j$  if and only if  $a \leq d_{ij}$  for all  $i, j < n$ . Let  $Q_{\bar{x}} = {}^n\{x_0, x_1, \dots, x_{n-1}\}$ . Consider the following equivalence relation  $\sim$  on  $Q_{\bar{x}}$ :*

$\bar{y} \sim \bar{z}$  if and only if  $\bar{z} = \tau|\bar{y}$  for some finite permutation  $\tau$ ,

$\bar{y}, \bar{z} \in Q_{\bar{x}}$ .

*Let us choose and fix representative tuples for the equivalence classes concerning  $\sim$  such that each representative tuple is of the form  $[i_0|j_0][i_1|j_1] \cdots [i_k|j_k]|\bar{x}$  for some  $k \geq 0, i_0, i_1, \dots, i_k, j_0, j_1, \dots, j_k < n$ . Such representative tuples exist. Indeed, for every  $\bar{y} \in Q_{\bar{x}}$ ,  $\exists \tau$  finite permutation,  $\exists k \geq 0, i_0, i_1, \dots, i_k, j_0, j_1, \dots, j_k < n$  such that*

$$\bar{y} = \tau|[i_0|j_0][i_1|j_1] \cdots [i_k|j_k]|\bar{x}.$$

*Let  $\bar{z} = \tau^{-1}|\bar{y}$ , then  $\bar{z} \sim \bar{y}$  and  $\bar{z} = [i_0|j_0][i_1|j_1] \cdots [i_k|j_k]|\bar{x}$ . Let  $\text{Rt}$  denote this fixed set of representative tuples. We define the transposition network  $T_{\bar{x}}^{(a)} : Q_{\bar{x}} \rightarrow \text{At}\mathfrak{A}$  as follows:*

- *If  $\bar{y} \in \text{Rt}$ , then  $\bar{y} = [i_0|j_0][i_1|j_1] \cdots [i_k|j_k]|\bar{x}$ , for some  $i_0, i_1, \dots, i_k, j_0, j_1, \dots, j_k < n$ . Let  $T_{\bar{x}}^{(a)}(\bar{y}) = (t_{i_0}^{j_0} \cdots t_{i_k}^{j_k})^{(\mathfrak{A}_{\text{pt}}\mathfrak{A})}a$ . This is well defined by Lemma 2.1.*
- *If  $\bar{z} = \sigma|\bar{y}$  for some finite permutation  $\sigma$  and some  $\bar{y} \in \text{Rt}$ , then let  $T_{\bar{x}}^{(a)}(\bar{z}) = \sigma T_{\bar{x}}^{(a)}(\bar{y})$ .*

**Lemma 2.3.** *The above definition is unique.*

*Proof.* The first part is well defined by Lemma 2.1. Now we need to prove that if  $\sigma|\bar{y} = \tau|\bar{y}$  for some finite permutations  $\sigma, \tau$  and some  $\bar{y} \in \text{Rt}$ , then  $s_\sigma T_{\bar{x}}^{(a)}(\bar{y}) = s_\tau T_{\bar{x}}^{(a)}(\bar{y})$ . First, we need the following:

**Claim .** *If  $\bar{y} = \tau|\bar{y}$  for some finite permutation  $\tau$  and some  $\bar{y} \in \text{Rt}$ , then*

$$T_{\bar{x}}^{(a)}(\bar{y}) = s_\tau T_{\bar{x}}^{(a)}(\bar{y}).$$

*Proof.* It suffices to show that if  $\bar{y} = [i, j]|\bar{y}$  for some  $i, j < n$  and some  $\bar{y} \in \text{Rt}$ , then  $T_{\bar{x}}^{(a)}(\bar{y}) = s_{[i, j]} T_{\bar{x}}^{(a)}(\bar{y})$ . For, suppose that  $\bar{y} = [i, j]|\bar{y}$  for some  $i, j < n$  and some  $\bar{y} \in \text{Rt}$ , then  $y_i = y_j$  and then  $T_{\bar{x}}^{(a)}(\bar{y}) \leq d_{ij}$  by Lemma 2.2. Hence by Lemma 1.1,  $T_{\bar{x}}^{(a)}(\bar{y}) = s_{[i, j]} T_{\bar{x}}^{(a)}(\bar{y})$ .  $\square$

Returning to our prove, assume that  $\sigma|\bar{y} = \tau|\bar{y}$  for some finite permutations  $\sigma, \tau$  and some  $\bar{y} \in \text{Rt}$ . Then  $(\tau^{-1}|\sigma)|\bar{y} = \bar{y}$  and so  $s_{(\tau^{-1}|\sigma)} T_{\bar{x}}^{(a)}(\bar{y}) = T_{\bar{x}}^{(a)}(\bar{y})$ . Therefore,  $s_{\tau^{-1}} s_\sigma T_{\bar{x}}^{(a)}(\bar{y}) = T_{\bar{x}}^{(a)}(\bar{y})$ . Hence,  $s_\sigma T_{\bar{x}}^{(a)}(\bar{y}) = s_\tau T_{\bar{x}}^{(a)}(\bar{y})$ , and we are done.  $\square$

**Lemma 2.4.**

1. *Let  $\mathfrak{A}$  be an atomic  $PTA_n$  and  $a \in \text{At}\mathfrak{A}$ . Let  $\bar{x}$  be any  $n$ -tuple of nodes such that  $x_i = x_j$  if and only if  $a \leq d_{ij}$  for all  $i, j < n$ . Then  $PT_{\bar{x}}^{(a)}$  is an atomic  $\mathfrak{A}$  network.*
2. *Let  $\mathfrak{A}$  be an atomic  $TEA_n$  and  $a \in \text{At}\mathfrak{A}$ . Let  $\bar{x}$  be any  $n$ -tuple of nodes such that  $x_i = x_j$  if and only if  $a \leq d_{ij}$  for all  $i, j < n$ . Then  $T_{\bar{x}}^{(a)}$  is an atomic  $\mathfrak{A}$  network.*

*Proof.* Straightforward from the above .  $\square$

**Lemma 2.5.** *Let  $\mathfrak{A} \in PTA_n \cup TEA_n$ . Then  $\exists$  can win any play of  $G_\omega(\mathfrak{A})$ .*

*Proof.* The proof is similar to that of Lemma 7.8 in [2]. Let  $\mathfrak{A}^+$  be the canonical extension of  $\mathfrak{A}$ . First note that  $\mathfrak{A}^+ \in TEA_n$  and  $\mathfrak{A}^+$  is atomic. Of course any  $\mathfrak{A}$  pre-network is an  $\mathfrak{A}^+$  pre-network. In each round  $t$  of the game  $G_\omega(\mathfrak{A})$ , where  $N_t$  is as above,  $\exists$  constructs an atomic  $\mathfrak{A}^+$  network  $M_t$  satisfying

$$M_t \supseteq N_t, \text{nodes}(M_t) = \text{nodes}(N_t), \text{edges}(M_t) = \text{edges}(N_t).$$

Then if  $\bar{x} \equiv_i \bar{y}$ , we have

$$N_t(\bar{x}) \cdot c_i N_t(\bar{y}) \geq M_t(\bar{x}) \cdot c_i M_t(\bar{y}) \neq 0.$$

$\exists$  starts by  $M_0 = N_0 = \emptyset$ . Suppose that we are in round  $t$  and assume inductively that  $\exists$  has managed to construct  $M_t \supseteq N_t$  as indicated above. We consider the possible moves of  $\forall$ .

- (1) Suppose that  $\forall$  picks a non zero element  $a \in \mathfrak{A}$ .  $\exists$  chooses an atom  $a^- \in \mathfrak{A}^+$  with  $a^- \leq a$ . She chooses new nodes  $x_0, \dots, x_{n-1}$  with  $x_i = x_j$  iff  $a^- \leq d_{ij}$ .

**If  $\mathfrak{A} \in PTA_n$ .** She creates two new relativized networks  $N_{t+1}, M_{t+1}$  with nodes those of  $N_t$  plus  $x_0, \dots, x_{n-1}$  and hyperedges those of  $N_t$  together with  $NSQ_{\bar{x}}$ . The new hyper labels in  $M_{t+1}$  are defined as follows:

$$M_{t+1} = M_t \cup PT_{\bar{x}}^{(a^-)}.$$

By Lemma 2.4 it follows that  $M_{t+1}$  is an atomic  $\mathfrak{A}^+$  network. Labels in  $N_{t+1}$  are given by

- $N_{t+1}(\bar{x}) = a \cdot \prod_{i,j:x_i=x_j} d_{ij}$ .
- $N_{t+1}(\bar{y}) = \prod_{i,j:y_i=y_j} d_{ij}$  for any other hyperedge  $\bar{y}$ .

**If  $\mathfrak{A} \in TEA_n$ .** She creates two new relativized networks  $N_{t+1}, M_{t+1}$  with nodes those of  $N_t$  plus  $x_0, \dots, x_{n-1}$  and hyperedges those of  $N_t$  together with  $Q_{\bar{x}}$ . The new hyper labels in  $M_{t+1}$  are defined as follows:

$$M_{t+1} = M_t \cup T_{\bar{x}}^{(a^-)}.$$

By Lemma 2.4 it follows that  $M_{t+1}$  is an atomic  $\mathfrak{A}^+$  network. Labels in  $N_{t+1}$  are given by

- $N_{t+1}(\tau|\bar{x}) = s_\tau a \cdot \prod_{i,j:x_i=x_j} d_{\tau i \tau j}$  for any finite permutation  $\tau$ .
- $N_{t+1}(\bar{y}) = \prod_{i,j:y_i=y_j} d_{ij}$  for any other hyperedge  $\bar{y}$ .

$\exists$  responds to  $\forall$ 's move in round  $t$  with  $N_{t+1}$ . One can check that  $N_t \subseteq N_{t+1} \subseteq M_{t+1}$ , as required.

- (2) If  $\forall$  picks an edge  $\bar{x}$  of  $N_t$  and an element  $a \in \mathfrak{A}$ ,  $\exists$  lets  $M_{t+1} = M_t$  and lets  $N_{t+1}$  be the same as  $N_t$  except that

**If  $\mathfrak{A} \in PTA_n$ .**  $N_{t+1}(\bar{x}) = N_t(\bar{x}) \cdot a$  if  $M_t(\bar{x}) \leq a$  and  $N_{t+1}(\bar{x}) = N_t(\bar{x}) \cdot -a$  otherwise. Because  $M_t(\bar{x})$  is an atom in  $\mathfrak{A}^+$ , it follows that if  $M_t(\bar{x}) \not\leq a$ , then  $M_t(\bar{x}) \leq -a$ , so this is satisfactory.

**If  $\mathfrak{A} \in TEA_n$ .** for every finite permutation  $\tau$ ,  $N_{t+1}(\tau|\bar{x}) = N_t(\tau|\bar{x}) \cdot s_\tau a$  if  $M_t(\tau|\bar{x}) \leq s_\tau a$  and  $N_{t+1}(\tau|\bar{x}) = N_t(\tau|\bar{x}) \cdot -s_\tau a$  otherwise. Because  $M_t(\tau|\bar{x})$  is an atom in  $\mathfrak{A}^+$  for every finite permutation  $\tau$ , it follows that if  $M_t(\tau|\bar{x}) \not\leq s_\tau a$ , then  $M_t(\tau|\bar{x}) \leq -s_\tau a$ , so this is satisfactory.

- (3) Alternatively  $\forall$  picks  $\bar{x} \in N_t$   $i < n$  and  $b \in \mathfrak{A}$  such that  $N_t(\bar{x}) \leq c_i b$ . Let  $M_t(\bar{x}) = a^-$ . If there is  $z \in M_t$  with  $M_t(\bar{x}_z^i) \leq b$  then we are done. In more detail,  $\exists$  lets  $M_{t+1} = M_t$  and define  $N_{t+1}$  accordingly. Else, there is no such  $z$ . We have  $c_i a^- \cdot b \neq 0$  (inside  $\mathfrak{A}^+$ ), indeed

$$\begin{aligned} c_i(c_i a^- \cdot b) &= c_i a^- \cdot c_i b \\ &= c_i(a^- \cdot c_i b) \\ &= c_i a^- \neq 0. \quad (\text{since } a^- \leq c_i b) \end{aligned}$$

Choose an atom  $b^- \in \mathfrak{A}^+$  with  $b^- \leq c_i a^- \cdot b$ . Then we have  $b^- \leq b$  and  $b^- \leq c_i a^-$ . But by Lemma 2.2 (iv) we also have  $a^- \leq c_i b^-$ .

**If  $\mathfrak{A} \in PTA_n$ .** Let  $G$  be the  $\mathfrak{A}^+$  network with nodes  $\{x_0, \dots, x_{n-1}, z\}$ ,  $z$  a new node, and the hyperedges are the sequences in  $NSQ_{\bar{x}} \cup NSQ_{\bar{t}}$  and  $G = PT_{\bar{x}}^{(a^-)} \cup PT_{\bar{t}}^{(b^-)}$  where  $\bar{t} = \bar{x}_z^i$ .

**If  $\mathfrak{A} \in TEA_n$ .** Let  $G$  be the  $\mathfrak{A}^+$  network with nodes  $\{x_0, \dots, x_{n-1}, z\}$ ,  $z$  a new node, and the hyperedges are the sequences in  $Q_{\bar{x}} \cup Q_{\bar{t}}$  and  $G = T_{\bar{x}}^{(a^-)} \cup T_{\bar{t}}^{(b^-)}$  where  $\bar{t} = \bar{x}_z^i$ .

Again this is well defined. Then, it easy to check that,  $M_t(i_1, \dots i_n) = G(i_1, \dots i_n)$  for all  $i_1, \dots i_n \in \text{Range}\bar{x}$ . That is the subnetworks of  $M_t$  and  $G$  with nodes  $\text{Range}\bar{x}$  are isomorphic. Then we can amalgamate  $M_t$  and  $G$  and define  $M_{t+1}$  as the outcome. The amalgamation here is possible, since the networks are only relativized, we don't have all hyperedges. By Lemma 2.4, one can check that  $M_{t+1}$  as so defined is an atomic  $\mathfrak{A}^+$  network. Now define  $N_{t+1}$  accordingly. That is  $N_{t+1}$  has the same nodes and edges of  $M_{t+1}$ , with labelling as for  $N_t$  except that

**If  $\mathfrak{A} \in PTA_n$ .**  $N_{t+1}(\bar{t}) = b$ . The rest of the other labels are defined to be

$$N_{t+1}(\bar{y}) = \prod_{i,j:y_i=y_j} d_{ij}.$$

**If  $\mathfrak{A} \in TEA_n$ .**  $N_{t+1}(\tau|\bar{t}) = s_\tau b$  for any finite permutaution  $\tau$ . The rest of the other labels are defined to be

$$N_{t+1}(\bar{y}) = \prod_{i,j:y_i=y_j} d_{ij}.$$

By Lemma 2.2, it is clear that  $N_t \subseteq N_{t+1} \subseteq M_{t+1}$  and then  $N_{t+1}$  is  $\mathfrak{A}$  network. Hence  $N_{t+1}$  is appropriate to be played by  $\exists$  in response to  $\forall$ 's move.

□

### 3 Main results

**Theorem 3.1.** *Let  $2 \leq n < \omega$ . If  $\mathfrak{A} \in PTA_n$ , then  $\mathfrak{A} \in ID_n$ .*

*Proof.* Let  $\mathfrak{A} \in PTA_n$ . We want to build an isomorphism from  $\mathfrak{A}$  to some  $\mathfrak{B} \in D_n$ . Consider a play  $N_0 \subseteq N_1 \subseteq \dots$  of  $G_\omega(\mathfrak{A})$  in which  $\exists$  plays as in the previous lemma and  $\forall$  plays every possible legal move. The outcome of the play is essentially a relativized representation of  $\mathfrak{A}$  defined as follows. Let  $N = \bigcup_{t < \omega} \text{nodes}(N_t)$ , and  $\text{edges}(N) = \bigcup_{t < \omega} \text{edges}(N_t) \subseteq {}^n N$ . By the definition of the networks,  $\wp(\text{edges}(N)) \in D_n$ . We make  $N$  into a representation by defining  $h : \mathfrak{A} \rightarrow \wp(\text{edges}(N))$  as follows

$$h(a) = \{\bar{x} \in \text{edges}(N) : \exists t < \omega (\bar{x} \in N_t \& N_t(\bar{x}) \leq a)\}.$$

$\forall$ -moves of the second kind guarantee that for any  $n$ -tuple  $\bar{x}$  and any  $a \in \mathfrak{A}$ , for sufficiently large  $t$  we have either  $N_t(\bar{x}) \leq a$  or  $N_t(\bar{x}) \leq -a$ . This ensures that  $h$  preserves the boolean operations.  $\forall$ -moves of the third kind ensure that the cylindrifications are respected by  $h$ . Preserving diagonals follows from the definition of networks. The first kind of  $\forall$ -moves tell us that  $h$  is one-one. But the construction of the game under consideration ensures that  $h$  is onto, too. In fact  $\bar{h}$  is a representation from  $\mathfrak{A}$  onto  $\mathfrak{B} \in D_n$ . This follows from the definition of networks.

□

**Theorem 3.2.** *Let  $2 \leq n < \omega$ . If  $\mathfrak{A} \in TEA_n$ , then  $\mathfrak{A} \in IDpe_n$ .*

*Proof.* Let  $\mathfrak{A} \in TEA_n$ . As above consider a play  $N_0 \subseteq N_1 \subseteq \dots$  of  $G_\omega(\mathfrak{A})$  in which  $\exists$  plays as in the previous lemma and  $\forall$  plays every possible legal move. The outcome of the play is essentially a relativized representation of  $\mathfrak{A}$  defined as follows. Let  $N = \bigcup_{t < \omega} nodes(N_t)$ , and  $edges(N) = \bigcup_{t < \omega} edges(N_t) \subseteq {}^n N$ . Again by the definition of the networks, it is easy to see that  $\wp(edges(N)) \in Dpe_n$ . We make  $N$  into a representation by defining  $h : \mathfrak{A} \rightarrow \wp(edges(N))$  as follows

$$h(a) = \{\bar{x} \in edges(N) : \exists t < \omega (\bar{x} \in N_t \& N_t(\bar{x}) \leq a)\}.$$

As the previous theorem,  $h$  preserves the boolean operations, the cylindrifications and the diagonals. Now we check transpositions. Let  $\bar{y} \in h(s_{ij}a)$ . Then there exists  $t < \omega$  such that  $N_t(\bar{y}) \leq s_{ij}a$ . Hence  $s_{ij}N_t(\bar{y}) = N_t([i, j]|\bar{y}) \leq a$ . The other inclusion is similar. The preservation of the substitutions follows directly from the preservation of the cylindrifications and the diagonals.  $\square$

**Theorem 3.3.** *Let  $2 \leq n < \omega$ . If  $\mathfrak{A} \in TA_n$ , then  $\mathfrak{A} \in IDp_n$ .*

*Proof.* It is enough to prove that every  $TA_\alpha$  is embeddable in a reduct of some  $TEA_\alpha$ . For, use the same method in the proof of Proposition 9. in [10], it uses only axioms  $(F_3)$ ,  $(F_4)$  and  $(F_6)$ . This method depend on the fact that  $\mathfrak{A}$  is definable by positive equations only. So  $\mathfrak{A}$  is canonical, and we can define diagonals in this canonical extension,  $\mathfrak{B}$  say, by  $d_{ij} = \bigcap \{y \in B : s_j^i y = 1\}$ , for every  $i, j \in \alpha$ . Then it can be shown that  $B$  with those constants is in  $TEA_\alpha$ .  $\square$

**Theorem 3.4.** *Let  $2 \leq n < \omega$ . If  $\mathfrak{A} \in SA_n$ , then  $\mathfrak{A} \in IDS_n$ .*

*Proof.* We will use the same analogue in the proof of Proposition 9. in [10]. Now, let  $2 \leq n < \omega$  and  $\mathfrak{A} \in SA_n$ . Then  $\mathfrak{A}$  is definable by positive equations only. Indeed, There is an axiomatization of Boolean algebra involving only meets and joins, so a Boolean homomorphism is specified by respecting meets and joins. By this we get rid of negation. Therefore by (I) in [3, p. 440],  $\mathfrak{A}$  is a subalgebra of a complete and atomic  $\mathfrak{B}$  such that  $\mathfrak{B} \models (C_0 - C_7)$ . Let  $d_{ij} = \bigcap \{y \in B : s_j^i y = 1\}$ , for every  $i, j \in \alpha$ . This definition is justified because  $\mathfrak{B}$  is complete. Our aim is to prove that  $\mathfrak{B}$  with this constants satisfies the axioms  $C_0 - C_7$  and this finishes the prove. For, it is enough to prove that  $\mathfrak{B} \models (C_5 - C_7)$ .

**Claim 3.1.** *For every  $i, j \in \alpha$  and every set  $K$ ,  $s_j^i(\bigcap_{k \in K} y_k) = \bigcap_{k \in K} s_j^i y_k$ .*

*Proof.* See [10, Claim 9.1.]  $\square$

**Claim 3.2.** *For every  $i, j \in \alpha$ ,  $\mathfrak{B} \models s_j^i d_{ij} = 1$ .*

*Proof.* Let  $i, j \in \alpha$ ,

$$\begin{aligned} \mathfrak{B} \models s_j^i d_{ij} &= s_j^i \{y \in B : s_j^i y = 1\} \\ &= \{s_j^i y : y \in B \text{ and } s_j^i y = 1\} \\ &= 1. \end{aligned}$$

$\square$

**Claim 3.3.** *For every  $i, j \in \alpha$  and every  $x \in B$ , if  $i \neq j$  then  $\mathfrak{B} \models s_j^i x = c_i(x \cdot d_{ij})$ .*

*Proof.* Let  $y \in B$  be such that  $s_j^i y = 1$ .

$$\begin{aligned} \mathfrak{B} \models s_j^i [-(x \cdot y) + s_j^i x] &= -(s_j^i x \cdot s_j^i y) + s_j^i s_j^i x \\ &= -s_j^i x + s_j^i x \\ &= 1, \end{aligned}$$

therefore,  $\mathfrak{B} \models x \cdot y \leq s_j^i x$ . Hence,

$$\begin{aligned} \mathfrak{B} \models \bigcap \{x \cdot y : y \in B \text{ and } s_j^i y = 1\} &\leq s_j^i x \\ \mathfrak{B} \models x \cdot d_{ij} &\leq s_j^i x, \end{aligned}$$

i.e.,  $\mathfrak{B} \models c_i(x \cdot d_{ij}) \leq s_j^i x$ . On the other direction,

$$\begin{aligned} \mathfrak{B} \models x \cdot d_{ij} &\leq c_i(x \cdot d_{ij}) \\ \mathfrak{B} \models s_j^i(x \cdot d_{ij}) &\leq s_j^i(c_i(x \cdot d_{ij})) \\ \mathfrak{B} \models s_j^i x \cdot s_j^i d_{ij} &\leq c_i(x \cdot d_{ij}) \\ \mathfrak{B} \models s_j^i x &\leq c_i(x \cdot d_{ij}). \end{aligned}$$

□

Now,

$$\begin{aligned} \mathfrak{B} \models d_{ij} &= \{y \in A : s_i^i y = 1\} \\ &= \{y \in A : y = 1\} \\ &= 1. \end{aligned}$$

$$\begin{aligned} \mathfrak{B} \models s_j^i [-(d_{ij} \cdot c_i(d_{ij} \cdot x)) + x] &= -(s_j^i d_{ij} \cdot s_j^i c_i(d_{ij} \cdot x)) + s_j^i x \\ &= -c_i(d_{ij} \cdot x) + s_j^i x \\ &\geq -s_j^i x + s_j^i x \\ &= 1. \end{aligned}$$

Therefore,  $\mathfrak{B} \models d_{ij} \cdot c_i(d_{ij} \cdot x) \leq x$ .

$$\begin{aligned} \mathfrak{B} \models s_j^i [-c_k(d_{ik} \cdot d_{kj}) + d_{ij}] &= s_j^i [-s_i^k d_{kj} + d_{ij}] \\ &= -s_j^i s_i^k d_{kj} + s_j^i d_{ij} \\ &= -s_j^i s_j^k d_{kj} + 1 \\ &= 1. \end{aligned}$$

Therefore,  $\mathfrak{B} \models c_k(d_{ik} \cdot d_{kj}) \leq d_{ij}$ . Hence we proved that  $\mathfrak{B} \models (C_0 - C_7)$  and also  $\mathfrak{B} \models s_j^i x = c_i(d_{ij} \cdot x)$ . This finishes the prove.

□

We have proved our theorems for finite dimensions. Now we turn to the infinite dimensional case. We give a general method of lifting representability for finite dimensional cases to the transfinite; that can well work in other contexts.

**Theorem 3.5.** *Assume that  $\alpha \geq \omega$ . Then  $PTA_\alpha = ID_\alpha$ ,  $TEA_\alpha = IDpe_\alpha$ ,  $TA_\alpha = IDp_\alpha$  and  $SA_\alpha = IDS_\alpha$ .*

*Proof.* We will consider the case of  $PTA_n$  and the other cases are similar. First, we know that  $PTA_n = ID_n$  for every finite  $n < \omega$ . We want to show that  $PTA_\alpha = ID_\alpha$  for any infinite  $\alpha$ . First note that :

1. For any  $\mathfrak{A} \in PTA_\alpha$  and  $\rho : n \rightarrow \alpha$ ,  $n \in \omega$  and  $\rho$  is one to one, define  $\mathfrak{Rd}^\rho \mathfrak{A}$  as in [3, Definition 2.6.1]. Then  $\mathfrak{Rd}^\rho \mathfrak{A} \in PTA_n$ .
2. For any  $n \geq 2$  and  $\rho : n \rightarrow \alpha$  as above,  $ID_n \subseteq S\mathfrak{Rd}^\rho ID_\alpha$  as in [4, Theorem 3.1.121].
3.  $ID_\alpha$  is closed under the ultraproducts, cf. [4, Lemma 3.1.90].

Now we show that if  $\mathfrak{A} \in PTA_\alpha$ , then  $\mathfrak{A}$  is representable. First, for any  $\rho : n \rightarrow \alpha$ ,  $\mathfrak{Rd}^\rho \mathfrak{A} \in PTA_n$ . Hence  $\mathfrak{Rd}^\rho \mathfrak{A}$  is in  $ID_n$  and so it is in  $S\mathfrak{Rd}^\rho ID_\alpha$ . Let  $J$  be the set of all finite one to one sequences with range in  $\alpha$ . For  $\rho \in J$ , let  $M_\rho = \{\sigma \in J : \rho \subseteq \sigma\}$ . Let  $U$  be an ultrafilter of  $J$  such that  $M_\rho \in U$  for every  $\rho \in J$ . Then for  $\rho \in J$ , there is  $\mathfrak{B}_\rho \in ID_\alpha$  such that  $\mathfrak{Rd}^\rho \mathfrak{A} \subseteq \mathfrak{Rd}^\rho \mathfrak{B}_\rho$ . Let  $\mathfrak{C} = \prod \mathfrak{B}_\rho / U$ ; it is in  $UpID_\alpha = D_\alpha$ . Define  $f : \mathfrak{A} \rightarrow \prod \mathfrak{B}_\rho$  by  $f(a)_\rho = a$ , and finally define  $g : \mathfrak{A} \rightarrow \mathfrak{C}$  by  $g(a) = f(a)/U$ . Then  $g$  is an embedding.  $\square$

**Remark.** Let  $\mathfrak{A}$  be an algebra in some class of our interest.  $\mathfrak{A}$  is said to be completely representable if there is a representation  $f : \mathfrak{A} \rightarrow P(V)$  such that  $\bigcup \{f(x) : x \text{ an atom}\} = V$ .<sup>6</sup> Therefore, according to the representations that are built in the proofs of Theorems 3.1, 3.2, 3.3, 3.4 and 3.5, every atomic algebra in  $PTA_\alpha \cup TEA_\alpha \cup TA_\alpha \cup SA_\alpha$  is completely representable.

Here we compare our classes with other important classes existing in the literature. Given a set  $U$  and a mapping  $p \in {}^\alpha U$ , then the set

$${}^\alpha U^{(p)} = \{x \in {}^\alpha U : x \text{ and } p \text{ are different only in finitely many places}\}$$

is called the weak space determined by  $p$  and  $U$ .

**Definition 3.1. Class  $Gw_\alpha$**  A set algebra in  $Crs_\alpha$  is called a generalized weak cylindric relativized set algebra if there are sets  $U_k$ ,  $k \in K$ , and mappings  $p_k \in {}^\alpha U_k$  such that  $V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$ , where  $V$  is the unit.

**Class  $Gwp_\alpha$  ( $Gwpe_\alpha$ )** A set algebra in  $Prs_\alpha$  ( $Pers_\alpha$ ) is called a generalized weak polyadic (equality) relativized set algebra if there are sets  $U_k$ ,  $k \in K$ , and mappings  $p_k \in {}^\alpha U_k$  such that  $V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$ , where  $V$  is the unit.

**Class  $Gws_\alpha$**  A set algebra in  $Srs_\alpha$  is called a generalized weak substitution relativized set algebra if there are sets  $U_k$ ,  $k \in K$ , and mappings  $p_k \in {}^\alpha U_k$  such that  $V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$ , where  $V$  is the unit.

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<sup>6</sup>A characterization of the completely representable algebras, cf. [8, Lemma 2.1]



A known characterization of the class  $Gwpe_\alpha$  is : If  $V$  is the unit of an  $A \in Prs_\alpha$ , then  $A \in Gwp_\alpha$  if and only if  $y \in V$  implies  $\tau|y \in V$  for every finite transformation  $\tau$ . Using this property one can prove that  $Gwpe_\alpha = Dpe_\alpha$ , so we can replace  $Dpe_\alpha$  by  $Gwpe_\alpha$  in Theorem 3.2 and Theorem 3.5. But the same is not true for the other types, for example, for finite  $n \in \omega$ , the class  $Gw_\alpha$  coincide with the class of locally square cylindric algebras. Andreka gave a finite schema axiomatization for the former class in [9], and Andreka's axioms and  $PTA_\alpha$  are not definitionally equivalent.

## 4 All varieties considered have the superamalgamation property

**Definition 4.1.** Let  $K$  be a class of algebras having a boolean reduct.  $\mathfrak{A}_0 \in K$  is in the amalgamation base of  $K$  if for all  $\mathfrak{A}_1, \mathfrak{A}_2 \in K$  and monomorphisms  $i_1 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_1$ ,  $i_2 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_2$  there exist  $D \in K$  and monomorphisms  $m_1 : \mathfrak{A}_1 \rightarrow D$  and  $m_2 : \mathfrak{A}_2 \rightarrow D$  such that  $m_1 \circ i_1 = m_2 \circ i_2$ . If in addition,  $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$  where  $\{j, k\} = \{1, 2\}$ , then we say that  $\mathfrak{A}_0$  lies in the super amalgamation base of  $K$ . Here  $\leq$  is the boolean order.  $K$  has the (super) amalgamation property ((SUP)AP), if the (super) amalgamation base of  $K$  coincides with  $K$ .

We now show using a result of Marx, that all varieties considered have the superamalgamation property (SUPAP). We consider  $TA_\alpha = ID_\alpha$ . The rest of the cases are the same. For a set  $V$ ,  $B(V)$  denotes the Boolean algebra  $(\wp(V), \cap, \sim)$ .

**Definition 4.2.** 1. A frame of type  $TA_\alpha$  is a first order structure

$$F = (V, C_i, S_i^j, S_{ij})_{i,j \in \alpha},$$

where  $V$  is an arbitrary set and  $C_i$ ,  $S_i^j$  and  $S_{ij}$  are binary relations for all  $i, j \in \alpha$ .

2. Given a frame  $F$ , its complex algebra denote by  $F^+$  is the algebra

$$(B(V), c_i, s_i^j, s_{ij})_{i,j},$$

where for  $X \subseteq V$ ,  $s_i^j(X) = \{s \in V : \exists t \in X, (t, s) \in s_i^j\}$ , and same for  $c_i$  and  $s_{ij}$ .

3. Given  $K \subseteq TA_\alpha$ , then  $Cm^{-1}K = \{F : F^+ \in K\}$ .

4. Given a family  $(F_i)_{i \in I}$  a zigzag product of these frames is a substructure of  $\prod_{i \in I} F_i$  such that the projection maps restricted to  $S$  are onto.

**Theorem 4.1.** (Marx) Assume that  $K$  is a canonical variety and  $L = Cm^{-1}K$  is closed under finite zigzag products. Then  $K$  has the superamalgamation property.

*Proof.* See [13, Lemma 5.2.6 p. 107. ]. □

**Theorem 4.2.** The variety  $TA_\alpha$  has SUPAP.

*Proof.* Since  $TA_\alpha$  is defined by positive equations then it is canonical. In this case  $L = Cm^{-1}TA_\alpha$  consists of frames  $(V, C_i, S_i^j, S_{i,j})$  such that if  $s \in V$ , then  $[i, j]|s \in V$  and  $[i|j]|s$  is in  $V$ . Moreover,  $(x, y) \in C_i$  iff  $x$  and  $y$  agree off  $i$ ,  $(x, y) \in S_i^j$  iff  $[i|j]|x = y$  and same for  $S_{ij}$ . The first order correspondants of the positive equations translated to the class of frames will be Horn formulas, hence clausifiable and so  $L$  is closed under finite zigzag products.  $\square$

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